Similarity reduction of the modified Yajima-Oikawa equation

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Abstract

We study a similarity reduction of the modified Yajima-Oikawa hierarchy. The hierarchy is associated with a non-standard Heisenberg subalgebra in the affine Lie algebra of type $A_2^{(1)}$. The system of equations for self-similar solutions is presented as a Hamiltonian system of degree of freedom two, and admits a group of Bäcklund transformations isomorphic to the affine Weyl group of type $A_2^{(1)}$. We show that the system is equivalent to a two-parameter family of the fifth Painlevé equation.

1 Introduction

In applications of the theory of affine Lie algebras to integrable hierarchies, the *Heisenberg subalgebras* play important roles, since they correspond to the varieties of time-evolutions. Let $\hat{\mathfrak{g}}$ be the untwisted affine Lie algebra associated with a finite-dimensional simple Lie algebra $\mathfrak{g}$. Up to conjugacy, the Heisenberg subalgebras in $\hat{\mathfrak{g}}$ are in one-to-one correspondence with the conjugacy classes of the Weyl group of $\mathfrak{g}$ [3]. In particular, the conjugacy class containing the Coxeter element, to which the *principal* Heisenberg subalgebra of $\hat{\mathfrak{g}}$ is associated, leads to the Drinfel'd-Sokolov hierarchy [2], whereas the class of the identity element corresponds to the *homogeneous* Heisenberg subalgebra. Associated with arbitrary conjugacy class, M. F. de Groot, T. J. Hollowood, J. L. Miramontes [1] developed the theory of integrable systems called generalized Drinfel'd-Sokolov hierarchies.

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When $\mathfrak{g}$ is of type A_{n-1}, the conjugacy classes are parametrized by the partitions of n. In this paper we consider the *modified Yajima-Oikawa hierarchy*, which turns out to be a hierarchy related to the affine Lie algebra of type $A_2^{(1)}$ and its non-standard Heisenberg subalgebra associated with the partition $(2,1)$, while the *principal* (resp. *homogeneous*) case corresponds to the partition (3) (resp. $(1,1,1)$).

Among the issues on integrable hierarchies, the study of similarity reduction is important. For example, M. Noumi and Y. Yamada introduced a higher order Painlevé system associated with the affine root system of type $A_{n-1}^{(1)}$ [7] and now the system is known to be equivalent to a similarity reduction of the system associated with the Coxeter class (n) of A_{n-1}. The aim of this paper is to investigate a similarity reduction of the modified Yajima-Oikawa hierarchy. Starting with universal viewpoints, we derive a system of ordinary differential equations for unknown functions $f_0, f_1, f_2, u_0, u_1, u_2, g, q, r$ and complex paremeters $\alpha_0, \alpha_1, \alpha_2$:

$$\begin{array}{ll}
\alpha_0' = \alpha_1' = \alpha_2' = 0, & \\
f_0' = f_0(u_2 - u_0) - \alpha_0, & g' = g(u_0 - u_2) - qf_1 + rf_2 + \alpha_0 + 4, \\
f_1' = f_1(u_0 - u_1) - r\alpha_1, & 3q' = 3q(u_1 - u_0) + qf_0 - f_2, \\
f_2' = f_2(u_1 - u_2) - q\alpha_2, & 3r' = 3r(u_2 - u_1) - rf_0 + f_1.
\end{array} \tag{1.1}$$

where $' = d/dx$ denote the derivative with respect to the independent variable x. Under the algebraic relations

$$\begin{gathered}
\alpha_0 + \alpha_1 + \alpha_2 = -4, \quad g = f_0 + 3qr, \quad u_0 + u_1 + u_2 = 0, \quad u_1 = qr, \\
2gu_0 = qf_1 - rf_2 - gqr - \alpha_0 - 2,
\end{gathered} \tag{1.2}$$

the system (1.1) turns out to be equivalent to the fifth Painlevé equation for $y = -f_0/(3u_1)$:

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(Ay + \frac{B}{y}\right) + \frac{C}{x}y + D\frac{y(y+1)}{(y-1)},$$

where the change of variable $x \to x^2$ is employed and the parameters are given by

$$A = \frac{1}{2}\left(\frac{\alpha_2 - \alpha_1}{12}\right)^2, \quad B = -\frac{1}{2}\left(\frac{\alpha_0}{4}\right)^2, \quad C = -\frac{\alpha_2 - \alpha_1}{18}, \quad D = -\frac{1}{18}.$$

On introducing the system (1.1), we shall describe the system in three ways:

1. Compatibility condition for a system of linear differential equations (Section 5),
2. A Hamiltonian system whose degree of freedom is two (Theorem 2),
3. Hirota bilinear equations for τ-functions (Theorem 3).

The system (1.1) has a symmetry of the affine Weyl group of type $A_2^{(1)}$ as a group of Bäcklund transformations. First we give the symmetry as the compatibility of gauge transformations of linear differential equations and state it in the automorphism of the differential field

$$K = \mathbf{C}(\alpha_0, \alpha_1, \alpha_2, f_0, f_1, f_2, g, q, r, u_0, u_1, u_2)$$

with the derivation $' : K \to K$ defined by (1.1) and algebraic relations (1.2) (Theorem 1). Then we extend the action of affine Weyl group on K to the extended field $\widehat{F}$ of K:

$$\widehat{F} = \mathbf{C}(\alpha_0, \alpha_1, \alpha_2, x; \tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2, \tau_0', \tau_1', \tau_2', \sigma_1', \sigma_2')$$

as a Bäcklund transformations, which is discussed in section 11 (Theorem 5).

The paper is organized as follows. In Sect.2, we review the notation related to the affine Lie algebra of type $A_2^{(1)}$. On the basis of the affine Lie algebra, we introduce the modified Yajima-Oikawa hierarchy in Sect.3. In Sect.4, we consider a condition of self-similarity on the solutions of the hierarchy. This condition yields a system of ordinary diferential equations, which is a main object in this paper. In Sect.5, the condition of self-similarity is also presented as a Lax-type equation. In Sect.6, we give a Weyl group symmetry of the system as a gauge transformation of the Lax equatoin (Theorem 1). In Sect.7 a Hamiltonian structure is introduced (Theorem 2). In Sect.8 we prove that our system is equivalent to a two-parameter family of the fifth Painlevé equation. In Sect.9 we introduce a set of τ-functions and give a bilinear form of differential system (Theorem 3). Then in Sect.10 we lift the action of Weyl group to the τ-functions (Theorem 4) and give a Jacobi-Trudi type formula (10.4) for the Weyl group orbit of the τ-functions. In Sect.11, we prove that the Weyl group action on the τ-functions commute with the derivation $' = d/dx$.

2 Preliminaries on the affine Lie algebra of type $A_2^{(1)}$

In this section, we collect necessary notions about the affine Lie algebra of type $A_2^{(1)}$. We mainly follow the notation used in [4], to which one should refer for further details.

Let $\mathfrak{g} = \mathfrak{sl}_3$. The affine Lie algebra $\hat{\mathfrak{g}}$ is realized as a central extension of the loop algebra $L\mathfrak{g} = \mathfrak{sl}_3(\mathbf{C}[z, z^{-1}])$, together with the derivation $d = z\partial_z$

$$\hat{\mathfrak{g}} = \mathfrak{sl}_3\left(\mathbf{C}[z, z^{-1}]\right) \oplus \mathbf{C}c \oplus \mathbf{C}d,$$

where c denotes the canonical central element. Let us define the Chevalley generators $E_i, F_i, H_i (i = 0, 1, 2)$ for the affine Lie algebra $\hat{\mathfrak{g}}$ by

$$E_0 = zE_{3,1},\ E_1 = E_{1,2},\ E_2 = E_{2,3},\ F_0 = z^{-1}E_{1,3},\ F_1 = E_{2,1},\ F_2 = E_{3,2}, \tag{2.1}$$

$$H_0 = c + E_{3,3} - E_{1,1}, \quad H_1 = E_{1,1} - E_{2,2}, \quad H_2 = E_{2,2} - E_{3,3},$$

where $E_{i,j}$ is the matrix unit $E_{i,j} = (\delta_{ia}\delta_{jb})_{a,b=1}^3$. The Cartan subalgebra of $\hat{\mathfrak{g}}$ is defined as $\hat{\mathfrak{h}} = \bigoplus_{i=0}^2 \mathbf{C}H_i \oplus \mathbf{C}d$. We introduce the simple roots α_j and the fundamental weights Λ_j as the following linear functionals on the Cartan subalgebra $\hat{\mathfrak{h}}$:

$$\langle H_i, \alpha_j \rangle = a_{ij}, \quad \langle H_i, \Lambda_j \rangle = \delta_{ij} \quad (i = 0, 1, 2), \quad \langle d, \alpha_j \rangle = \delta_{0j}, \quad \langle d, \Lambda_j \rangle = 0$$

for $j = 0, 1, 2$, where $(a_{ij})_{i=0}^3$ is the generalized Cartan matrix of type $A_2^{(1)}$ defined by

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

We define a non degenerate symmetric bilinear form $(\,.\,|\,.\,)$ on $V = \hat{\mathfrak{h}}^*$ as follows:

$$(\alpha_i|\alpha_j) = a_{ij}, \quad (\alpha_i|\Lambda_0) = \delta_{i0}, \quad (\Lambda_0|\Lambda_0) = 0.$$

We define simple reflections s_i $(i = 0, 1, 2)$ by

$$s_i(\lambda) = \lambda - \langle H_i, \lambda\rangle \,\alpha_i, \quad \lambda \in V.$$

They satisfy the fundamental relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (i = 0, 1, 2),$$

where the indices are understood as elements of $\mathbf{Z}/3\mathbf{Z}$. Consider the group

$$W = \langle s_0, s_1, s_2\rangle \subset \mathrm{GL}(V), \tag{2.2}$$

generated by the simple reflections. The group W is called the affine Weyl group of type $A_2^{(1)}$.

3 Modified Yajima-Oikawa hierarchy

In this section we introduce the modified Yajima-Oikawa hierarchy as generalized Drinfel'd-Sokolov reduction associated to the loop algebra $L\mathfrak{g} = \mathfrak{sl}_3\left(\mathbf{C}[z, z^{-1}]\right)$, following [1]. Let us introduce the following derivation on $L\mathfrak{g}$:

$$D = 4z\frac{\partial}{\partial z} - \mathrm{diag}(-1, 0, 1). \tag{3.1}$$

Set

$$L\mathfrak{g}_j = \{A \in L\mathfrak{g} \mid [D, A] = jA\}.$$

Then we have a $\mathbf{Z}$-gradation $L\mathfrak{g} = \oplus_j L\mathfrak{g}_j$. Note that

$$\deg(E_0) = -\deg(F_0) = 2, \quad \deg(E_j) = -\deg(F_j) = 1 \quad (j = 1, 2).$$

Consider the particular element

$$\gamma = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{bmatrix}$$

and let $\mathfrak{s}$ be the centralizer of γ in $L\mathfrak{g}$

$$\mathfrak{s} = \mathrm{Ker}\,(\mathrm{ad}\gamma) = \{A \in L\mathfrak{g} \mid [\gamma, A] = 0\}.$$

The subalgebra $\mathfrak{s}$ is a maximal commutative subalgebra in $\mathfrak{g}$, which has the following basis:

$$\gamma_{4j+2} = z^j\gamma, \quad \gamma_{4j} = z^j \mathrm{diag}(1, -2, 1) \quad (j \in \mathbf{Z}).$$

Then $\mathfrak{s}$ is a graded subalgebra of $L\mathfrak{g}$ with respect to the gradation. We have $\gamma_{2j} \in L\mathfrak{g}_{2j}$. The commutative subalgebra $\mathfrak{s}$ is the image of a Heisenberg subalgebra in $\hat{\mathfrak{g}}$ associated with the conjugacy class $(2,1)$ ([3], see also [10] and [5]). We put $\mathfrak{b} := \oplus_{j\geq 0} L\mathfrak{g}_j$.

To introduce our hierarchy, we begin with the differential operator

$$L := \frac{\partial}{\partial x} - \gamma - Q,$$

where Q is an x-dependent element of $\mathfrak{b}_{<2}$. We set $\mathfrak{s}^\perp := \mathrm{Im}\,(\mathrm{ad}\gamma)$. It is clear that $\mathfrak{s}^\perp = \oplus_j \mathfrak{s}_j^\perp$, where $\mathfrak{s}_j^\perp := \mathfrak{s}^\perp \cap L\mathfrak{g}_j$. There is a unique formal series $U = \sum_{j=1}^\infty U_{-j}$ $(U_{-j} \in \mathfrak{s}_{-j}^\perp)$ such that the operator $L_0 := e^{\mathrm{ad}U}(L)$ has the form

$$L_0 = \frac{\partial}{\partial x} - \gamma - \sum_{j=0}^{\infty} h_{-2j}, \quad h_{-2j} \in \mathfrak{s}_{-2j}.$$

Moreover U_{-j} and h_{-2j} are polynomials in the components of Q and their x derivatives. For any $j > 0$ we set

$$B_{2j} = \left(e^{-\mathrm{ad}U}\gamma_{2j}\right)_{\geq 0}.$$

The modified Yajima-Oikawa hierarchy is defined by the Lax equations

$$\frac{\partial L}{\partial t_{2j}} = [B_{2j}, L] \quad (j = 1, 2, \ldots).$$

We describe the above construction concretely. First we set

$$Q = \begin{bmatrix} u_0 & r & 0 \\ 0 & u_1 & q \\ 0 & 0 & u_2 \end{bmatrix}, \quad u_0 + u_1 + u_2 = 0$$

and solve for the first few terms of U_j and h_j:

$$\begin{aligned}
U_{-1} =& -qE_{2,1} + rE_{3,2},\\
U_{-2} =& \frac{u_2 - u_0}{4}(z^{-1}E_{1,3} - E_{3,1}),\\
U_{-3} =& \left[\left(\frac{3u_0}{8} + \frac{3u_1}{2} - \frac{3u_2}{8} - qr\right) r + r'\right] E_{1,2}\\
&+ \left[\left(\frac{7u_0}{8} - u_1 + \frac{u_2}{8} + qr\right) q + q'\right] E_{2,3},\\
U_{-4} =& \left[\frac{u_0' - u_2'}{8} + \frac{q'r + 3qr'}{8} + \left(\frac{u_0}{16} - \frac{5u_1}{16} + \frac{u_2}{16} + \frac{5}{16}qr\right) qr\right] (E_{1,1} - E_{3,3}),
\end{aligned}$$

$$\begin{aligned}
h_0 =& \frac{qr - u_1}{2}\gamma_0,\\
h_{-2} =& \left[\frac{u_0^2 + u_2^2}{8} - \frac{u_0 u_2}{4} - \frac{q'r + 3qr'}{4} - \left(\frac{u_0}{8} - \frac{5u_1}{8} + \frac{u_2}{8} + \frac{3}{8}qr\right) qr\right] \gamma_{-2}.
\end{aligned}$$

Here $'$ means $\partial/\partial x$. In fact, h_0 is a constant along all the flows and we can put $h_0 = 0$ (see [1]). So we fix

$$u_1 = qr \tag{3.2}$$

from now on. By using U_j's and condition (3.2) we have

$$B_2 = \gamma_2 + \begin{bmatrix} u_0 & r & 0 \\ 0 & u_1 & q \\ 0 & 0 & u_2 \end{bmatrix}, \tag{3.3}$$

$$B_4 = \gamma_4 + 3 \begin{bmatrix} -qr' + qru_2 & r' - ru_2 & 0 \\ qz & qr' - q'r + qru_1 & -q' - qu_0 \\ -qrz & rz & q'r + qru_0 \end{bmatrix} \tag{3.4}$$

The modified Yajima-Oikawa equation is obtained by the following zero-curvature condition:

$$\frac{\partial B_2}{\partial t_4} = \frac{\partial B_4}{\partial t_2} - [B_2, B_4]. \tag{3.5}$$

In fact this yields the following system of differential equations:

$$q_t + 3\left(q'' + q(-qr' + u_0' + qru_2 + u_2^2)\right) = 0, \tag{3.6}$$
$$r_t - 3\left(r'' - r(-q'r + u_2' - qru_0 + u_0^2)\right) = 0, \tag{3.7}$$

$$(u_0)_t = 3(-qr' + qru_2)', \quad (u_1)_t = 3(qr' - q'r + qru_1)', \quad (u_2)_t = 3(q'r + qru_0)'. \tag{3.8}$$

Here we identify x and t_2, and put $t = t_4$.

Remark: This system of equations is related to the Yajima-Oikawa equation [11]:

$$\Psi_t + 3\left(\Psi'' + u\Psi\right) = 0, \tag{3.9}$$
$$\Phi_t - 3\left(\Phi'' + u\Phi\right) = 0, \tag{3.10}$$
$$u_t + 6(\Psi\Phi)' = 0. \tag{3.11}$$

The relation is established by the following map, which takes a solution q, r, u_j $(j = 0, 1, 2)$ of (3.6) (3.7), (3.8) into a solution Ψ, Φ, u of (3.9), (3.10), (3.11) and is an analog of the Miura map in the case of KdV and mKdV equations:

$$\Psi = -q' - qu_0, \quad \Phi = r' - ru_2, \quad -u = u_0^2 + u_2^2 + u_0 u_2 + u_0' + qr'.$$

4 Similarity reduction

In this section we consider a self-similarity condition on the solutions of the modified Yajima-Oikawa equation (3.6), (3.7), (3.8). These are the main object of this paper. A solution $q(x,t)$, $r(x,t)$, $u_j(x,t)$ $(j=0,1,2)$ is said to be self-similar if

$$q(\lambda^2 x, \lambda^4 t) = \lambda^{-1} q(x,t), \quad r(\lambda^2 x, \lambda^4 t) = \lambda^{-1} r(x,t), \quad u_j(\lambda^2 x, \lambda^4 t) = \lambda^{-2} u_j(x,t). \tag{4.1}$$

Here we count a degree of variables by $\deg x = \deg t_2 = -2$, $\deg t = \deg t_4 = -4$. Note that such functions are uniquely determined by its values at fixed t, say at $t = 1/4$. Differentiating (4.1) with respect to λ at $\lambda = 1$, we obtain the Euler equations

$$2x\frac{\partial q}{\partial x} + 4t\frac{\partial q}{\partial t} = -q, \quad 2x\frac{\partial r}{\partial x} + 4t\frac{\partial r}{\partial t} = -r, \quad 2x\frac{\partial u_j}{\partial x} + 4t\frac{\partial u_j}{\partial t} = -2u_j.$$

At $t = 1/4$ these identities become

$$\frac{\partial q}{\partial t} = -2\frac{\partial (xq)}{\partial x} + q, \quad \frac{\partial r}{\partial t} = -2\frac{\partial (xr)}{\partial x} + r, \quad \frac{\partial u_j}{\partial t} = -2\frac{\partial (xu_j)}{\partial x}.$$

This can be written in the matrix form

$$\frac{\partial B_2}{\partial t} = -2\frac{\partial (xB_2)}{\partial x} + [D, B_2],$$

where D is the derivation defined in (3.1). Substituting this last identity into the zero-curvature equation (3.5), we obtain

$$\frac{\partial M}{\partial x} = \left[4z\frac{\partial}{\partial z} - M, B_2\right], \tag{4.2}$$

where we set

$$M = \begin{bmatrix} \varepsilon_1 & f_1 & g \\ 0 & \varepsilon_2 & f_2 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} + z\begin{bmatrix} 1 & 0 & 0 \\ 3q & -2 & 0 \\ f_0 & 3r & 1 \end{bmatrix} := \operatorname{diag}(-1,0,1) + 2xB_2 + B_4. \tag{4.3}$$

The correspondence of variables is given as follows: v

$$\varepsilon_1 = -1 + 2xu_0 - 3q(r' - ru_2), \tag{4.4}$$
$$\varepsilon_2 = 2xu_1 + 3(qr' - q'r + qru_1), \tag{4.5}$$
$$\varepsilon_3 = 1 + 2xu_2 + 3r(q' + qu_0) \tag{4.6}$$

and $g = 2x$,

$$f_0 = 2x - 3qr, \quad f_1 = 2xr + 3(r' - ru_2), \quad f_2 = 2xq - 3(q' + qu_0). \tag{4.7}$$

Here we regard the variables $q = q(x,1/4)$, $r = r(x,1/4)$, $u_j = u_j(x,1/4)$ $(j=0,1,2)$ are functions only in x. Note that the definition of M has a freedom of adding a constant diagonal matrix and here we normalize

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0. \tag{4.8}$$

5 Lax pair formalism

Consider the following system of linear differential equations for the column vector $\overrightarrow{\psi} = {}^t(\psi_1, \psi_2, \psi_3)$ of three unknown functions $\psi_i = \psi_i(z, x)$ $(i = 1, 2, 3)$:

$$4z\frac{\partial}{\partial z}\overrightarrow{\psi} = M\overrightarrow{\psi}, \quad \frac{\partial}{\partial x}\overrightarrow{\psi} = B\overrightarrow{\psi}. \tag{5.1}$$

We assume that the matrix M is (4.3) and $B = B_2$ (3.3) where the variables $\varepsilon_j, f_j, u_j, q, r$ and g are functions in x. Then the compatibility condition of system (5.1)

$$\left[4z\frac{\partial}{\partial z} - M, \frac{\partial}{\partial x} - B\right] = 0 \tag{5.2}$$

is equivalent to the relations

$$\begin{array}{ll} \varepsilon_1' = \varepsilon_2' = \varepsilon_3' = 0, & g = f_0 + 3qr, \\ f_0' = f_0(u_2 - u_0) - (\varepsilon_3 - \varepsilon_1 - 4), & g' = g(u_0 - u_2) - qf_1 + rf_2 - \varepsilon_1 + \varepsilon_3, \\ f_1' = f_1(u_0 - u_1) - r(\varepsilon_1 - \varepsilon_2), & 3q' = 3q(u_1 - u_0) + qf_0 - f_2, \\ f_2' = f_2(u_1 - u_2) - q(\varepsilon_2 - \varepsilon_3), & 3r' = 3r(u_2 - u_1) - rf_0 + f_1. \end{array} \tag{5.3}$$

If we forget the relation (4.3) of M and B_1, B_2 and start from the Lax equation (5.3), we can recover some of the relations of variables. For instance, differentiating both-hand side of $g = f_0 + 3qr$ and eliminate the variables except g' by means of (5.3), we get $g' = 2$ and therefore assume

$$g = 2x.$$

In what follows we shall impose the following constraint on the variables:

$$u_0 + u_1 + u_2 = 0, \quad u_1 = qr. \tag{5.4}$$

The joint system (5.2) and (5.4) is the main object that we investigate in this paper. Using system (5.3) together with the constraint, we can derive the following equation:

$$2gu_0 = qf_1 - rf_2 - gqr - \varepsilon_3 + \varepsilon_1 + 2. \tag{5.5}$$

After the elimination of the variables f_0, u_0, u_1, u_2 by (5.3), (5.4) and (5.5), we obtain a system of ODE for the unknown functions f_1, f_2, q, r with the parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$. We can obtain the set of explicit formulae of f_1', f_2', q', r' in terms of f_1, f_2, q, r and g, and the results are

$$f_1' = \frac{f_1}{2g}(f_1q - f_2r) - \frac{3}{2}f_1qr + (\varepsilon_1 - \varepsilon_3)\frac{f_1}{2g} - (\varepsilon_1 - \varepsilon_2)r + \frac{f_1}{g}, \tag{5.6}$$

$$f_2' = \frac{f_2}{2g}(f_1q - f_2r) + \frac{3}{2}f_2qr + (\varepsilon_1 - \varepsilon_3)\frac{f_2}{2g} - (\varepsilon_2 - \varepsilon_3)q + \frac{f_2}{g}, \tag{5.7}$$

$$q' = -\frac{q}{2g}(f_1q - f_2r) + \frac{q^2r}{2} - (\varepsilon_1 - \varepsilon_3)\frac{q}{2g} + \frac{gq - f_2}{3} - \frac{q}{g}, \tag{5.8}$$

$$r' = -\frac{r}{2g}(f_1q - f_2r) - \frac{qr^2}{2} - (\varepsilon_1 - \varepsilon_2)\frac{r}{2g} - \frac{gr - f_1}{3} - \frac{r}{g}. \tag{5.9}$$

In Sect.7 we present the system of ODE in the Hamiltonian form.

Remark. Using (5.5) and (5.3), we can also derive the following differential equation:

$$gu_0' = (\varepsilon_2 - \varepsilon_3)qr + \frac{f_2}{3}(rf_0 - f_1) - 2u_0. \tag{5.10}$$

6 Bäcklund transformations

Let us pass to the investigation of a group of Bäcklund transformations. For this purpose, it is convenient to introduce the following set of parameters:

$$\alpha_0 = \varepsilon_3 - \varepsilon_1 - 4, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3. \tag{6.1}$$

They are identified with the simple roots of the affine root system of type $A_2^{(1)}$.

We define the Bäcklund transformations for the system by considering the gauge transformations of the linear system (5.1)

$$s_i\overrightarrow{\psi} = G_i\overrightarrow{\psi} \quad (i = 0, 1, 2). \tag{6.2}$$

The matrices G_i are given as follows:

$$G_i = 1 + \frac{\alpha_i}{f_i}F_i \quad (i = 0, 1, 2), \tag{6.3}$$

where F_0, F_1, F_2 are Chevalley generators (2.1) of the loop algebra $\mathfrak{sl}_3(\mathbf{C}[z, z^{-1}])$. The compatibility condition of (5.1) and (6.2) is

$$s_i(M) = G_iMG_i^{-1} + 4z\frac{\partial G_i}{\partial z}G_i^{-1}, \quad s_i(B) = G_iBG_i^{-1} + \frac{\partial G_i}{\partial x}G_i^{-1}. \tag{6.4}$$

On the components of the matrices M, B, the actions of $s_i (i = 0, 1, 2)$ are given explicitly as in the following tables:

	f_0	f_1	f_2	g	q	r
s_0	f_0	$f_1 + 3r\frac{\alpha_0}{f_0}$	$f_2 - 3q\frac{\alpha_0}{f_0}$	g	q	r
s_1	$f_0 - 3r\frac{\alpha_1}{f_1}$	f_1	$f_2 + g\frac{\alpha_1}{f_1}$	g	$q + \frac{\alpha_1}{f_1}$	r
s_2	$f_0 + 3q\frac{\alpha_2}{f_2}$	$f_1 - g\frac{\alpha_2}{f_2}$	f_2	g	q	$r - \frac{\alpha_2}{f_2}$

	α_0	α_1	α_2	u_0	u_1	u_2
s_0	$-\alpha_0$	$\alpha_1 + \alpha_0$	$\alpha_2 + \alpha_0$	$u_0 + \frac{\alpha_0}{f_0}$	u_1	$u_2 - \frac{\alpha_0}{f_0}$
s_1	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$	$u_0 - r\frac{\alpha_1}{f_1}$	$u_1 + r\frac{\alpha_1}{f_1}$	u_2
s_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	u_0	$u_1 - q\frac{\alpha_2}{f_2}$	$u_2 + q\frac{\alpha_2}{f_2}$

The automorphisms $s_i (i=0,1,2)$ generate a group of Bäcklund transformations for our differential system. To state this fact clearly, it is convenient to introduce the field

$$K = \mathbf{C}(\alpha_0, \alpha_1, \alpha_2, f_0, f_1, f_2, g, q, r, u_0, u_1, u_2), \tag{6.5}$$

where the generators satisfy the following algebraic relations:

$$\alpha_0 + \alpha_1 + \alpha_2 = -4, \quad f_0 = g - 3qr, \quad u_0 + u_1 + u_2 = 0, \quad u_1 = qr,$$

$$2gu_0 = qf_1 - rf_2 - gqr - \varepsilon_3 + \varepsilon_1 + 2.$$

We have the automorphisms s_i $(i=0,1,2)$ of the field K defined by the above table. Note that the field K is thought to be a differential field with the derivation $' : K \to K$ defined by (5.3).

Theorem 1 *The automorphism s_0, s_1, s_2 of K define a representation of the affine Weyl group W (2.2) on the field K such that the action of the each element $w \in W$ commutes with the derivation of the differential field K.*

Theorem 1 is proved by straightforward computations. Note that the independent variable $x = g/2$ is fixed under the action of W.

7 Hamiltonian structure

We shall equip K (6.5) with the Poisson algebra structure $\{\ ,\ \} : K \times K \to K$ defined as follows:

$\{\ ,\ \}$	f_1	f_2	q	r
f_1	0	g	1	0
f_2	$-g$	0	0	-1
q	-1	0	0	0
r	0	1	0	0

That is, $\{f_1, f_2\} = g$ and so on. Note that the Poisson structure comes from the Lie algebra structure of $\hat{\mathfrak{g}}$ (see [9] for an exposition). We can describe the action of s_i $(i=0,1,2)$ on the generators $f = f_j, u_j, q, r, g$ $(j=0,1,2)$ of K by

$$s_i(f) = f + \frac{\alpha_i}{f_i}\{f_i, f\}.$$

We introduce the function h by

$$\begin{aligned} h :=& \frac{1}{2}(f_1 q^2 r + f_2 q r^2) - \frac{1}{4g}(f_1^2 q^2 + f_2^2 r^2 + q^2 r^2 g^2) + \left(\frac{qr}{2g} - \frac{1}{3}\right) f_1 f_2 \\ &+ \left(\frac{g}{3} - \frac{\alpha_1 + \alpha_2}{2g}\right) f_1 q + \left(\frac{g}{3} + \frac{\alpha_1 + \alpha_2}{2g}\right) f_2 r - \left(\frac{g}{3} - \frac{\alpha_1 - \alpha_2}{2g}\right) qrg. \end{aligned}$$

Then the differential system (5.6)–(5.9) can be expressed

$$\begin{aligned} f_1' &= \{h, f_1\} + \frac{f_1}{g}, \quad q' = \{h, q\} - \frac{q}{g}, \\ f_2' &= \{h, f_2\} + \frac{f_2}{g}, \quad r' = \{h, r\} - \frac{r}{g}. \end{aligned} \tag{7.1}$$

Let us introduce the variables

$$p_1 = f_1, \quad q_1 = q, \quad p_2 = \frac{f_2}{g} - q, \quad q_2 = -gr.$$

It is easy to show that

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0 \quad (i, j = 1, 2).$$

Theorem 2 *Let H be the function defined as*

$$\begin{aligned} xH = &-\frac{1}{4}p_1p_2q_1q_2 - \frac{1}{8}\left(p_1^2q_1^2 + p_2^2q_2^2\right) - \frac{1}{2}p_1q_1^2q_2 \\ &- \frac{1}{4}\left(\alpha_1 + \alpha_2 + 2\right)p_1q_1 - \frac{1}{4}\left(\alpha_1 + \alpha_2 - 2\right)p_2q_2 - \frac{\alpha_1}{2}q_1q_2 - \frac{2x^2}{3}(q_2 + p_1)p_2 \end{aligned}$$

Then the system of ODEs (5.6), (5.7), (5.8), (5.9) is equivalent to the Hamiltonian system

$$\frac{dq_1}{dx} = \frac{\partial H}{\partial p_1}, \quad \frac{dq_2}{dx} = \frac{\partial H}{\partial p_2}, \quad \frac{dp_1}{dx} = -\frac{\partial H}{\partial q_1}, \quad \frac{dp_2}{dx} = -\frac{\partial H}{\partial q_2}. \tag{7.2}$$

Proof. We define

$$H = h - \frac{f_1q + f_2r}{g} + qr$$

and rewrite this in the coordinate p_j, q_j ($j = 1, 2$). Then the equations (7.1) can be expressed as (7.2). □

The behavior of the Hamiltonian under the Bäcklund transformations is given by the simple formulae

$$s_0(\tilde{H}) = \tilde{H} + 6qr\frac{\alpha_0}{f_0}, \quad s_j(\tilde{H}) = \tilde{H} \ (j = 1, 2),$$

where we set $\tilde{H} = xH + a$ with the correction term

$$a = \frac{1}{24}(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2 - 4).$$

8 Reduction to the fifth Painlevé equation

In this section, we show the system (5.3) is equivalent to a two-parameter family of the fifth Painlevé equation. By linear change of the independent variable, we ensure the normalization

$$f_0 + \frac{f_1}{r} + \frac{f_2}{q} + 3\left(\frac{q'}{q} - \frac{r'}{r}\right) = 3g = 6x \tag{8.1}$$

holds. After the elimination of u_0 and u_2, we have

$$f_0' = -\frac{f_0}{3}\left(\frac{f_1}{r} - \frac{f_2}{q}\right) + \frac{f_0 u_1'}{u_1} - \alpha_0 \tag{8.2}$$

$$\left(\frac{f_1}{r}\right)' = -\frac{f_1}{3r}\left(\frac{f_2}{q} - f_0 + \frac{3u_1'}{u_1}\right) - \alpha_1, \tag{8.3}$$

$$\left(\frac{f_2}{q}\right)' = -\frac{f_2}{3q}\left(f_0 - \frac{f_1}{r} + \frac{3u_1'}{u_1}\right) - \alpha_2. \tag{8.4}$$

Here we introduce a new variable

$$y := -\frac{f_0}{3u_1}.$$

Notice the relations

$$y - 1 = -\frac{2x}{3u_1}, \quad \frac{y'}{y-1} = \frac{1}{x} - \frac{u_1'}{u_1} \tag{8.5}$$

holds by $f_0 = g - 3qr = 2x - 3u_1$. Then we rewrite (8.2) as

$$y' = -\frac{y}{3}\left(\frac{f_1}{r} - \frac{f_2}{q}\right) + \frac{\alpha_0}{3u_1}, \tag{8.6}$$

After differentiating (8.6), elimination of the variables f_1, f_2, q, r, u_1 by (8.1), (8.3), (8.4), (8.5), (8.6) and the definition of the constant ε_2 (4.5) leads to the following equation of y:

$$\begin{aligned} y'' = &\left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{x} + \frac{(y-1)^2}{8x^2}\left(\varepsilon_2^2 y - \frac{\alpha_0^2}{y}\right) \\ &- \frac{2x^2 y}{9} - \frac{4x^2 y}{9(y-1)} - \frac{(\alpha_2 - \alpha_1)y}{3} + \frac{\varepsilon_2 y}{3}. \end{aligned} \tag{8.7}$$

We put $\xi = x^2$, then the equation (8.7) can be brought into the fifth Painlevé equation

$$y_{\xi\xi} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y_\xi)^2 - \frac{1}{\xi}y_\xi + \frac{(y-1)^2}{\xi^2}\left(Ay + \frac{B}{y}\right) + \frac{C}{\xi}y - \frac{y(y+1)}{18(y-1)},$$

where

$$A = \frac{\varepsilon_2^2}{32}, \quad B = -\frac{\alpha_0^2}{32}, \quad C = -\frac{\varepsilon_2}{6}.$$

Note that $\varepsilon_2 = (\alpha_2 - \alpha_1)/3$ holds by (4.8) and (6.1).

9 τ-functions

We introduce the τ-functions τ_0,τ_1,τ_2, σ_1 and σ_2 to be the dependent variables satisfying the following equations:

$$\frac{f_1}{r} = 2x + 3\left(\frac{\sigma_2'}{\sigma_2} - \frac{\tau_0'}{\tau_0}\right), \quad \frac{f_2}{q} = 2x - 3\left(\frac{\sigma_1'}{\sigma_1} - \frac{\tau_0'}{\tau_0}\right), \quad q = -\frac{\sigma_1}{\tau_1}, \quad r = \frac{\sigma_2}{\tau_2}. \tag{9.1}$$

To fix the freedom of overall multiplication by a function in the defining equation (9.1) for τ_0, τ_1, τ_2, σ_1 and σ_2, we impose the equation

$$\begin{aligned}\left(\log \tau_0^2\tau_1^2\tau_2^2\sigma_1\sigma_2\right)'' + u_0^2 + u_2^2 + \left(u_0 - \frac{f_1}{3r} + \frac{2x}{3}\right)^2 + \left(u_2 - \frac{f_2}{3q} + \frac{2x}{3}\right)^2 \\ - \frac{2x}{9}\left(4x - \frac{f_1}{r} - \frac{f_2}{q}\right) - \frac{\alpha_1 - \alpha_2}{9} = 0.\end{aligned} \tag{9.2}$$

The differential equations for the variables q and r in the system (5.3) lead to

$$u_0 = \frac{\tau_1'}{\tau_1} - \frac{\tau_0'}{\tau_0}, \quad u_2 = \frac{\tau_0'}{\tau_0} - \frac{\tau_2'}{\tau_2} \tag{9.3}$$

respectively. Here we have used the relations

$$u_1 = qr = -\frac{\sigma_1\sigma_2}{\tau_1\tau_2}, \quad f_0 = 2x - 3qr = 2x + 3\frac{\sigma_1\sigma_2}{\tau_1\tau_2}.$$

If the equations (9.3) are satisfied, we have

$$u_1 = \frac{\tau_2'}{\tau_2} - \frac{\tau_1'}{\tau_1}, \tag{9.4}$$

by $u_0 + u_1 + u_2 = 0$ and therefore have the following formula of the variable f_0 in terms of the τ-functions:

$$f_0 = 2x + 3\left(\frac{\tau_1'}{\tau_1} - \frac{\tau_2'}{\tau_2}\right). \tag{9.5}$$

Let D_x and D_x^2 be Hirota's bilinear operators:

$$D_x F \cdot G := F'G - FG', \quad D_x^2 F \cdot G := F''G - 2F'G' + FG''.$$

In this notation, the relation $u_1 = qr$, for example, can be written in

$$D_x\tau_1 \cdot \tau_2 = \sigma_1\sigma_2. \tag{9.6}$$

We introduce a system of bilinear equations that leads to our differential system (5.3).

Theorem 3 *Let $\tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2$ be a set of functions that satisfies the following system of Hirota bilinear equations*:

$$\left(3D_x^2 - 2xD_x + \frac{1}{6}(\alpha_0 - 4\alpha_1 - 2)\right)\tau_0 \cdot \tau_1 = 0, \tag{9.7}$$

$$\left(3D_x^2 - 2xD_x - \frac{1}{6}(\alpha_0 - 4\alpha_2 - 2)\right)\tau_2 \cdot \tau_0 = 0, \tag{9.8}$$

$$\left(3D_x^2 - 2xD_x + \frac{1}{6}(\alpha_1 - \alpha_2 + 6)\right)\tau_1 \cdot \sigma_2 = 0, \tag{9.9}$$

$$\left(3D_x^2 - 2xD_x + \frac{1}{6}(\alpha_1 - \alpha_2 - 6)\right)\sigma_1 \cdot \tau_2 = 0, \tag{9.10}$$

together with (9.6). If we define the functions f_0, f_1, f_2, q, r, u_0, u_1 and u_2 by the formulae (9.1), (9.3), (9.4) then this set of functions satisfies our ODE system (5.3) together with algebraic equations (5.4).

Proof. We can verify that the differential equations for q and r are satisfied if we assume the existence of the τ-functions such that equations (9.1), (9.3) holds. The differential equations for f_0 is written as

$$3(g_1'' - g_2'') + 2 = (3(g_1' - g_2') + 2x)(2g_0' - g_1' - g_2') - \alpha_0, \tag{9.11}$$

where $g_j = \log \tau_j$, $(j = 0, 1, 2)$. This equation is obtained if we subtract (9.7) from (9.8). The differential equations for f_1 and f_2 can be rewritten as

$$\left(\frac{f_1}{r}\right)' = \frac{f_1}{r}\left(u_0 - u_1 - \frac{r'}{r}\right) - \alpha_1 \quad \left(\frac{f_2}{q}\right)' = \frac{f_2}{q}\left(u_1 - u_2 - \frac{q'}{q}\right) - \alpha_2, \tag{9.12}$$

respectively. In terms of the τ-functions, these equations read

$$3(h_2'' - g_0'') + 2 = (3(h_2' - g_0') + 2x)(2g_1' - g_0' - h_2') - \alpha_1, \tag{9.13}$$

$$3(g_0'' - h_1'') + 2 = (3(g_0' - h_1') + 2x)(2g_2' - g_0' - g_1') - \alpha_2, \tag{9.14}$$

where $h_1 = \log \sigma_1, h_2 = \log \sigma_2$. In fact, from (9.7) and (9.9) we can eliminate g_1'' to obtain (9.13). In the similar way from (9.8) and (9.10), we can eliminate g_2'' to obtain (9.14). □

We remark that the normalization of τ-functions (9.2) is obtained by taking the sum of four equations in this theorem.

10 Jacobi-Trudi type formula

In this section we lift the action of W to the τ-functions. Consider the field extension $\widetilde{K} = K(\tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2)$. Then we can prove the next Theorem by a direct computation.

Theorem 4 *We extend each automorphism s_i of K to an automorphism of the field $\widetilde{K} = K(\tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2)$ by the formulae $s_i(\tau_j) = \tau_j$ $(i \neq j)$, $s_i(\sigma_k) = \sigma_k$ $(i \neq k)$ and*

$$s_0(\tau_0) = f_0 \frac{\tau_2 \tau_1}{\tau_0}, \quad s_1(\tau_1) = f_1 \frac{\tau_0 \tau_2}{\tau_1}, \quad s_1(\sigma_1) = -(f_1 q + \alpha_1) \frac{\tau_0 \tau_2}{\tau_1}, \tag{10.1}$$

$$s_2(\tau_2) = f_2 \frac{\tau_1 \tau_0}{\tau_2}, \quad s_2(\sigma_2) = (f_2 r - \alpha_2) \frac{\tau_1 \tau_0}{\tau_2}. \tag{10.2}$$

Then these automorphisms define a representation of W on $\widetilde{K}$.

Following [6], we will describe the Weyl group orbit of the τ-functions (see also [9]). For any $w \in W$ and $k = 0, 1, 2$, there exists a rational function $\phi_w^{(k)} \in K$ such that

$$w(\tau_k) = \phi_w^{(k)} \prod_{i=0,1,2} \tau_i^{(\alpha_i | w(\Lambda_k))}. \tag{10.3}$$

We shall give an expression of $\phi_w^{(k)}$ in terms of the Jacobi-Trudi type determinant.

A subset M of $\mathbf{Z}$ is called a Maya diagram if $M \cap \mathbf{Z}_{\geq 0}$ and $M^c \cap \mathbf{Z}_{<0}$ are finite sets. We define an integer

$$c(M) := \sharp\,(M \cap \mathbf{Z}_{\geq 0}) - \sharp\,(M^c \cap \mathbf{Z}_{<0})$$

called the charge of M. If $c(M) = r$, we can express M as $\{i_k | k < r\}$ by using an strictly increasing sequence $i_k\,(k < r)$ such that $i_k = k$ for $k \ll r$. Then we associate a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ given by

$$\lambda_j = i_{r-j+1} - (r - j + 1), \quad (j = 1, 2, \ldots).$$

The Weyl group $W = \langle s_0, s_1, s_2 \rangle$ can be realized as a subgroup of the group of bijections $w : \mathbf{Z} \to \mathbf{Z}$ by setting

$$s_k = \prod_{j \in \mathbf{Z}} \sigma_{3j+k-1} \quad (k = 0, 1, 2),$$

where σ_i $(i \in \mathbf{Z})$ is the adjacent transposition $(i, i+1)$. For a Maya diagram M and $w \in W$, we see that $w(M) \subset \mathbf{Z}$ is also a Maya diagram of the same charge.

For any $w \in W$ and $k = 0, 1, 2$, let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the partition corresponding to the Maya diagram $M = w(\mathbf{Z}_{<k})$. We set

$$N_\lambda^{(k)} = \prod_{\substack{i<j \\ i \in M^c, j \in M}} (\varepsilon_i - \varepsilon_j),$$

where we impose the relation $\varepsilon_i - \varepsilon_{i+3} = -4$ $(i \in \mathbf{Z})$, so we have $N_\lambda^{(k)} \in \mathbf{C}[\alpha_0, \alpha_1, \alpha_2]$. We can apply the following formula due to Y.Yamada [12]:

$$\phi_w(\Lambda_k) = N_\lambda^{(k)} \det \left(g_{\lambda_j - j + i}^{(k-i+1)} \right)_{1 \leq i,j \leq r}. \tag{10.4}$$

Here $g_p^{(k)}$ $(k \in \mathbf{Z}/3\mathbf{Z}, p \in \mathbf{Z}_{>0})$ are the determinant of $p \times p$ matrix described as follows. First we define $g_p^{(0)}$ by

$$g_p^{(0)} := \frac{1}{N_p^{(0)}} \begin{vmatrix} f_{00} & f_{01} & f_{02} & & & 0 \\ \beta_1 & f_{11} & f_{12} & f_{13} & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & f_{p-3,p-1} \\ & & & \beta_{p-2} & f_{p-2,p-2} & f_{p-2,p-1} \\ 0 & & & & \beta_{p-1} & f_{p-1,p-1} \end{vmatrix},$$

where the components are

$$\begin{aligned} f_{i,i} &= f_i \quad (f_{i+3} = f_i), \\ f_{i,i+1} &= g\,(i \equiv 1), \quad 3q\,(i \equiv 2), \quad 3r\,(i \equiv 0), \\ f_{i,i+2} &= 1\,(i \equiv 1, 0), \quad -2\,(i \equiv 2), \end{aligned}$$

and $\beta_j = \sum_{i=j}^{p-1} \alpha_i = \varepsilon_j - \varepsilon_p$. Then we put $g_p^{(1)} = \pi(g_p^{(0)})$ and $g_p^{(2)} = \pi^2(g_p^{(0)})$ by the automorphism π:

$$\pi(f_{ij}) = f_{i+1,j+1}, \quad \pi(\varepsilon_j) = \varepsilon_{j+1}.$$

The formula (10.4) is valid since the action of $W = \langle s_0, s_1, s_2 \rangle$ in our setting is reduced from the action of A_∞ (cf. [9]):

$$s_i(\alpha_i) = -\alpha_i, \quad s_i(\alpha_{i\pm1}) = \alpha_{i\pm1} + \alpha_i, \quad s_i(\alpha_j) = \alpha_j \ (j \neq i, i \pm 1),$$

where $\alpha_j := \varepsilon_j - \varepsilon_{j+1}$ $(j \in \mathbf{Z})$ and

$$s_k(f_{i,j}) = f_{i,j} + (\delta_{k+1,i} f_{k,j} - \delta_{j,k} f_{i,k+1}) \frac{\alpha_k}{f_k}.$$

11 Differential field of τ-functions

In this section we give supplementary discussions on the affine Weyl group action. In particular, we consider a differential field of τ-functions that naturally contains the fields K and $\widetilde{K}$. The field $\widehat{F}$ we consider can be presented as

$$\mathbf{C}(\alpha_0, \alpha_1, \alpha_2, x; \tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2, \tau_0', \tau_1', \tau_2', \sigma_1', \sigma_2') \tag{11.1}$$

with some relations discussed below. Then the set of bilinear equations in Theorem 3 makes $\widehat{F}$ into the differential field. To show some basic facts on $\widehat{F}$, we introduce some intermediate fields.

Let F denote the extended field of $\mathbf{C}(\alpha_0, \alpha_1, \alpha_2, x)$ obtained by adjoining the variables g_0', g_1', g_2', h_1', h_2' with the following relations:

$$3(g_0' - 2h_2' + h_1')(g_1' - g_2') + 2x(g_0' - 2g_1' + g_2') + \alpha_1 + 1 = 0, \tag{11.2}$$
$$3(h_2' - 2h_1' + g_0')(g_1' - g_2') + 2x(g_1' - 2g_2' + g_0') + \alpha_2 + 1 = 0. \tag{11.3}$$

As in the proof of Theorem 3, we will identify g_j' with $(\log \tau_j)'$ and h_1', h_2' with $(\log \sigma_1)'$, $(\log \sigma_2)'$ respectively. Note that the relations (11.2), (11.3) correspond to (4.4), (4.5), (4.6). It is easy to see $F = \mathbf{C}(\alpha_0, \alpha_1, \alpha_2, x)(g_0', g_1', g_2')$, and g_0', g_1', g_2' are algebraically independent over $\mathbf{C}(\alpha_0, \alpha_1, \alpha_2, x)$. So if we fix $g_j'' \in F$ $(j = 0, 1, 2)$ in an arbitrary way, then we have a derivation on F. Now we want to introduce a derivation on F in such a way that is consistent with the bilinear equations. Actually we can prove the following lemma by lengthy but straightforward computations:

Lemma 1 *There exists a unique derivation on F such that the set of bilinear equations in Theorem 3 holds.*

Consider the extended field $\widehat{F} := F(\tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2)$ with a relation

$$\tau_1'\tau_2 - \tau_2\tau_1' = \sigma_1\sigma_2.$$

We can naturally extend the derivation by $\tau_j' = g_j'\tau_j$, $\sigma_k' = h_k'\sigma_k$ $(j = 0, 1, 2, k = 1, 2)$. Then we have the previous presentation (11.1). Now the next lemma is a direct consequence of Theorem 3.

Lemma 2 *We have a natural embedding of the differential fields*

$$K \subset \widehat{F}.$$

Our next task is to extend the affine Weyl group action on $\widetilde{K} = K(\tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2)$ (Theorem 4) to $\widehat{F}$. The following two lemmas can be easily verified.

Lemma 3 *By the following formulae, we can introduce an action of the affine Weyl group W on $\widehat{F}$ as a group of automorphisms:*

$$\begin{aligned}
&\frac{s_0(\tau_0')}{s_0(\tau_0)} = \frac{\tau_0'}{\tau_0} - \frac{\alpha_0}{f_0}, \\
&\frac{s_1(\tau_1')}{s_1(\tau_1)} = \frac{\tau_1'}{\tau_1} - \frac{\alpha_1}{f_1}\frac{\sigma_2}{\tau_2}, \quad \frac{s_1(\sigma_1')}{s_1(\tau_1)} = \frac{\sigma_1'}{\tau_1} - \frac{\alpha_1}{f_1}\frac{\tau_0'}{\tau_0}, \\
&\frac{s_2(\tau_2')}{s_2(\tau_2)} = \frac{\tau_2'}{\tau_2} + \frac{\alpha_2}{f_2}\frac{\sigma_1}{\tau_1}, \quad \frac{s_2(\sigma_2')}{s_2(\tau_2)} = \frac{\sigma_2'}{\tau_2} - \frac{\alpha_1}{f_1}\frac{\tau_0'}{\tau_0},
\end{aligned}$$

and $s_i(\tau_j') = \tau_j'$ $(i \neq j)$, $s_i(\sigma_k') = \sigma_k'$ $(i \neq k)$. Moreover this action is an extension of the action of W on $\widetilde{K}$.

Lemma 4 *For $i, j = 0, 1, 2$ and $k = 1, 2$ we have*

$$s_i(\tau_j') = s_i(\tau_j)', \quad s_i(\sigma_k') = s_i(\sigma_k)'.$$

Remark. Although we have introduced the Weyl group action on the τ-functions in an ad hoc manner, these formulae can be derived systematically by using the gauge matrices G_i (6.3), if we identify the τ-functions with the components of a *dressing matrix*. We will give an explanation of this point in a separate article.

The goal of this section is the following fact:

Theorem 5 *The derivation of $\widehat{F}$ commutes with the action of W on $\widehat{F}$.*

A straightforward verification of this fact may require quite a bit of calculations, because the second derivatives of τ-functions are determined implicitly by the bilinear equations. To avoid the complexity, we make use of the fact $\widehat{F} = \widetilde{K}(k)$, which is easily seen from (9.1), (9.3), and (9.4), where we set

$$k = 2\left(\frac{\tau_0'}{\tau_0} + \frac{\tau_1'}{\tau_1} + \frac{\tau_2'}{\tau_2}\right) + \frac{\sigma_1'}{\sigma_1} + \frac{\sigma_2'}{\sigma_2}.$$

As for the first derivatives of τ-functions, we have already lemma 4. Therefore, in order to prove Theorem 5, it suffices to show the next lemma.

Lemma 5

$$s_i(k') = s_i(k)' \quad (i = 0, 1, 2). \tag{11.4}$$

Proof. By Lemma 3, we have

$$s_0(k) - k = 2\left(\frac{s_0(\tau_0')}{s_0(\tau_0)} - \frac{\tau_0'}{\tau_0}\right) = -2\frac{\alpha_0}{f_0},$$

$$s_1(k) - k = 2\left(\frac{s_1(\tau_1')}{s_1(\tau_1)} - \frac{\tau_1'}{\tau_1}\right) + \left(\frac{s_1(\sigma_1')}{s_1(\sigma_1)} - \frac{\sigma_1'}{\sigma_1}\right), \tag{11.5}$$

$$s_2(k) - k = 2\left(\frac{s_2(\tau_2')}{s_2(\tau_2)} - \frac{\tau_2'}{\tau_2}\right) + \left(\frac{s_2(\sigma_2')}{s_2(\sigma_2)} - \frac{\sigma_1'}{\sigma_1}\right). \tag{11.6}$$

We can rewrite the right hand sides of (11.5) and (11.6) into

$$s_1(k) - k = -2\frac{\alpha_1}{f_1}r - \frac{\alpha_1(2xq - f_2)}{3q(f_1q + \alpha_1)},$$

$$s_2(k) - k = -2\frac{\alpha_2}{f_2}q - \frac{\alpha_2(2xr - f_1)}{3r(f_2r - \alpha_2)}$$

by using (9.1), (10.1) and (10.2). On the other hand, the normalization condition (9.2) reads

$$\begin{aligned} k' = &- u_0^2 - u_2^2 - \left(u_0 - \frac{f_1}{3r} + \frac{2x}{3}\right)^2 - \left(u_2 - \frac{f_2}{3q} + \frac{2x}{3}\right)^2 \\ &+ \frac{2x}{9}\left(4x - \frac{f_1}{r} - \frac{f_2}{q}\right) + \frac{\alpha_1 - \alpha_2}{9}. \end{aligned}$$

Then we can verify (11.4) by applying (6.4) to $s_i(k')$ and the ODE (5.3) to $s_i(k)'$. $\square$

12 Discussion

We have derived a two-parameter family of the fifth Painlevé equation as a similarity reduction of the modified Yajima-Oikawa hierarchy, which is related to a non-standard Heisenberg subalgebra of $A_2^{(1)}$. The system admits a group of Bäcklund transformations of type $W(A_2^{(1)})$. By a suitable modification of our construction, it may be possible to recover a *missing* parameter and get the fifth Painlevé with the full symmetry of type $W(A_3^{(1)})$. Combinatorial and/or representation theoretical structure of the hierarchy is also deserves to be investigated. A combinatorial aspect of representation associated with the Yajima-Oikawa hierarchy is studied by S. Leidwanger in [5]. It seems that the work is closely related some family of polynomial solutions of the fifth Painlevé equation. We hope that we discuss these issues in future publications.

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